Groups in Circle Puzzles

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Circle puzzles are sequential move puzzles that consist of intersecting rotating disks in the plane. Previous authors recognise the possibilities for diverse group-theoretic structure within this class of puzzles, but few have made progress in classifying all groups that can appear in circle puzzles. This paper discusses several methods that could help reach such a classification, including canonical decompositions into smaller groups, computational search for empirical classification, and a new algorithm for finding circle puzzle representations of particular sets of permutations.

1 Introduction

CIRCLE puzzles are a class of sequential move puzzles in the plane that consist of intersecting disks. Each move involves rotating a disk by some fixed increment. As with other sequential move puzzles, the goal is to use these moves to restore a scrambled puzzle to a canonical starting state. An example circle puzzle is shown in Figure 1. The left disk has two stops, and the right disk has three stops. Available moves are to rotate these disks in increments of π and $\frac{2\pi}{3}$ radians, respectively. The goal of the puzzle is to restore the current state with three separately coloured segments. Engel [1] offers a more detailed overview, highlighting the history and geometric properties of circle puzzles.

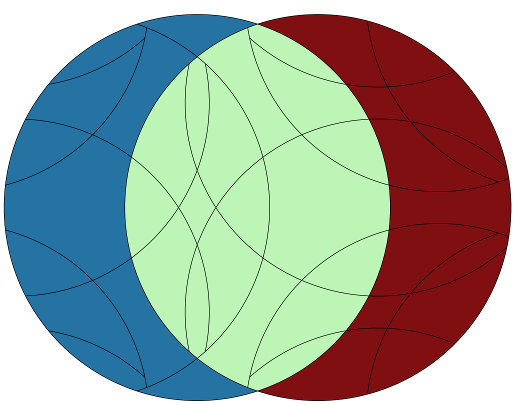

Figure 1. An example circle puzzle.

Like other sequential move puzzles, one can associate with each circle puzzle a group of permutations on moved pieces to precisely describe the set of reachable positions. Nevertheless, relative to other classes of puzzles, the group-theoretic nature of circle puzzles is poorly understood. For example, Wilson [2] showed that with a small number of exceptions, a class of sliding permutation puzzles embedded in graphs can only generate symmetric groups or alternating groups. This class of puzzles includes the famous 15 Puzzle. Scherphuis [3] and Yang [4] extended Wilson's result to another type of graph-embedded puzzles that loosely correspond to circle puzzles with small overlap between disks.

Montenegro *et al.* [5] analysed the group structure of puzzles based on rotating squares arranged in a grid. A move on such a puzzle consists of rotating a block of squares by 90 degrees. The authors showed that with some small exceptions, most groups in these puzzles also decompose into symmetric or alternating groups.

In contrast, little is known about the more general circle puzzles. Eidswick [6] observes that circle puzzle groups are generated by permutations of equal-length cycles, and that cycles belonging to two different generators share at most two moved pieces. However, he notes that these conditions are not sufficient for a particular set of generators to have a corresponding circle puzzle, and also notes that there is little research on permutation groups satisfying these properties.

This article presents several separate but complementary paths for studying circle puzzle groups. First, it discusses canonical ways of classifying circle puzzle groups by decomposing them into smaller groups in Section 2. In Section 3, the article discusses how to identify groups in a particular circle puzzle by computing a permutation representation, and notes several patterns that might lead to a general classification. Finally, the article introduces in Section 4 an algorithm for deciding whether a particular group can be realised by a (slightly more general) circle puzzle, strengthening Eidswick's results.

This article assumes some basic familiarity with group theory; a glossary of the terms used is provided in the appendix.

2 Decomposing Groups

2.1 Simple Groups

As with many other mathematical objects, groups are often best studied by breaking them up into

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