From Mathematical Proof to Puzzle

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This short note describes how an investigation into the construction of a geometric figure inadvertently led to the design of a published game, demonstrating that inspiration for games can occur where you least expect it.

1 Introduction

THE Al-Quds Star, shown in Figure 1, is a symbol found on a number of emblems and flags, based on the Muslim symbol Rub El Hizb. In Arabic, *rub* means one fourth or quarter, while *hizb* means a group or party.[1] This figure is also called Solomon's Star or Gadeiro. In my home country of Spain, it can be found on pavements, walls and ornaments in many regions, and it is the official symbol for the city of Teruel.

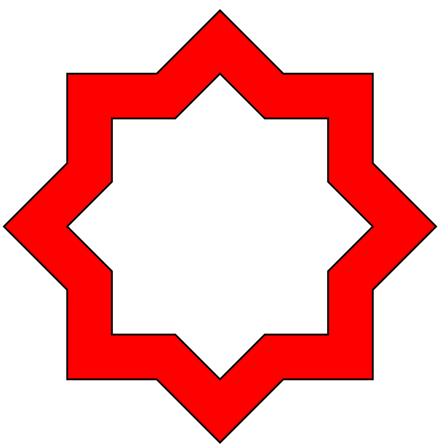

Figure 1. The Al-Quds Star or Gadeiro.

I was fascinated by this shape as soon as I saw it, and wanted to explore its geometry. This paper describes how this exploration led to the creation of new puzzle game.

2 Geometric Construction

Figure 2 shows the construction of the Gadeiro. It is based on two square frames, one of which is rotated 45° to the other, then their combined inner area is removed to give the final shape. Each square frame is defined by an inner and outer square, and the width of the frames is defined by the shape itself, as the corners of the smaller inner squares coincide with the edge midpoints of the larger outer squares (marked *a* in the Figure).

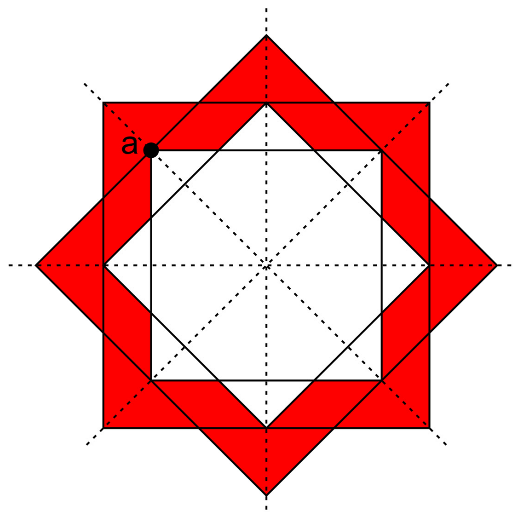

Figure 2. Construction of the Gadeiro figure.

Once I understood this construction, an interesting question arose: *is the area of the inner (white) shape equal to the area of the outer (red) shape?*

2.1 Proof by Geometry

First, note that the ratio of distances of rotated to non-rotated corners is $\sqrt{2}$:1, as shown in Figure 3.

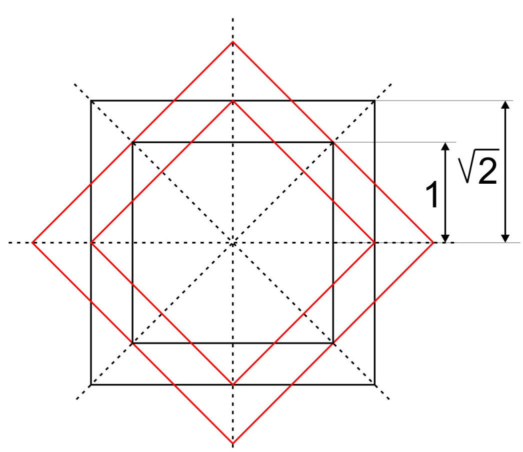

Figure 3. Key ratio of distances.

[1] https://en.wikipedia.org/wiki/Rub_el_Hizb

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Since $(\sqrt{2})^2 = 2$, this suggests that the total area enclosed by the outer shape is twice the area of the inner shape, hence the inner (white) and outer (red) shapes both have the same area. I prove this in a more detailed document elsewhere.[2]

Another geometrical proof is achieved by observing that the Gadeiro shape fits exactly inside a copy of itself when shrunk to half its area. Continuing this halving series, as per Figure 4, gives the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ which approaches 1 as the number of halvings approaches infinity:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \tag{5}$$

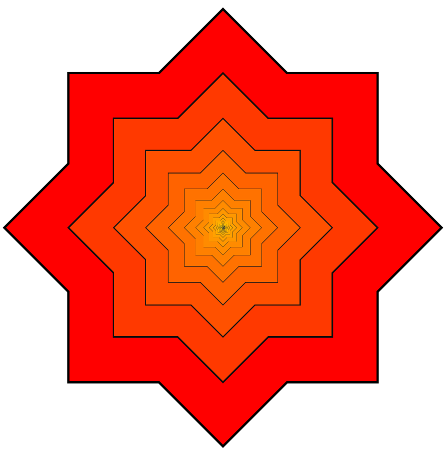

Figure 4. A halving series fills the inner area.

This manifestation of Zeno's famous Dichotomy Paradox [1] demonstrates that the areas of the inner and outer shapes are equal, but is very much a theoretical proof as such a series can never be physically constructed. Could there be a more direct and practical way to prove this equality?

2.2 Proof by Game

I then considered a more direct approach: *could the outer shape be dissected into pieces and reassembled to fill its interior?* Figure 5 shows the best dissection that I have come up with, which I believe could be optimal for this problem. The best dissection will have the fewest number of cuts giving the fewest number of pieces, and I liked the symmetry of the pieces produced.

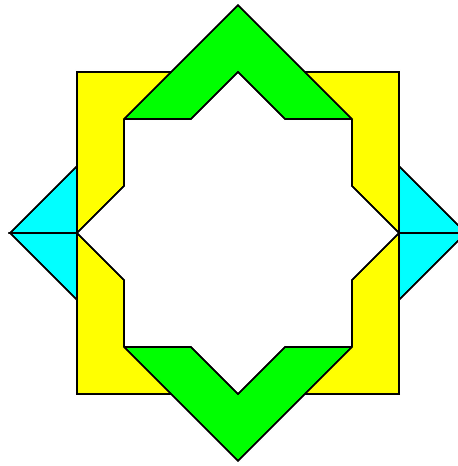

Figure 5. Dissection of the Gadeiro into pieces.

Figure 6 shows how these pieces can be reassembled to fill the inner shape, demonstrating by example that the inner and outer shape areas are equal.

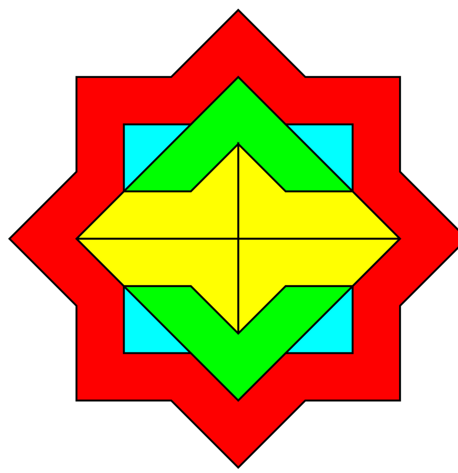

Figure 6. Pieces repacked to fill the inner shape.

3 A Game is Born

Apart from providing a nice proof of inner/outer area equality, these pieces have found a life of their own as a puzzle game called Gadeiro, which has now been published [2]. This is a form of Tangram puzzle [3] in which players use the pieces to form given silhouette shapes, such as those shown in Figure 7. Figure 8 shows a solution for one of these challenges.

This exercise shows how games can emerge from the most unlikely inspirations!

[2] http://www.gapdjournal.com/issues/issue-3-1/gadeiro.pdf

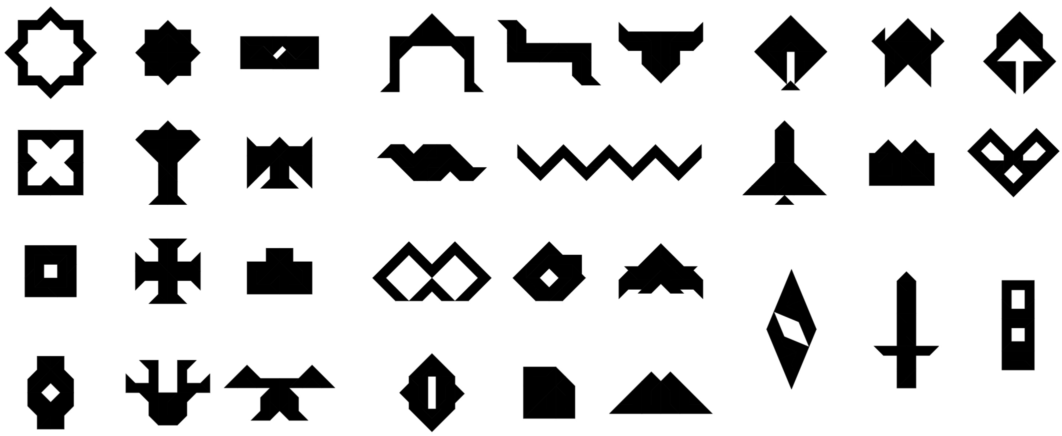

Figure 7. Some Gadeiro challenge shapes.

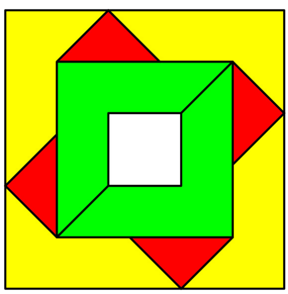

Figure 8. Solution of a Gadeiro challenge.

Acknowledgements

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References

[1] Salmon, W. (ed.), *Zeno's Paradoxes*, New York, Bobs-Merrill, 1970.

[2] N. Romeral Andrés, *Gadeiro*, Nestorgames, 2016. http://nestorgames.com/#gadeiro_detail

[3] Slocum, J., *The Tangram Book*, New York, Sterling, 2003.

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Gadeiro Challenge #6

Pack the pieces on the right to fill the shape on the left.