Limping Boards for Games

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This paper describes how the simple mathematical notion of the limping triangle may be extended to a broader class of limping polygons, and the implications this has for the design of game boards. Limping rectangular boards (with a square basis) and limping hexagonal boards (with a regular hexagonal basis) are shown to give the designer control over some useful game design parameters.

1 Introduction

A *limping triangle* is a right triangle in which the two short sides (i.e. not the hypotenuse) differ in length by exactly one unit [1, p. 116]. Figure 1 shows a limping triangle with short sides of length n and $n+1$.

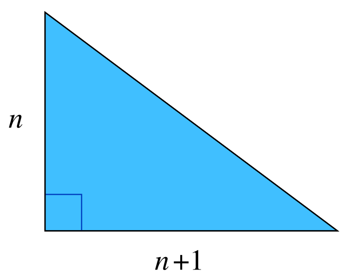

Figure 1. A limping triangle.

This paper describes how this concept can be extended to describe certain subsets of game boards. While the limping property does not confer any particularly interesting mathematical properties to the triangle, its use in the context of board shape can give certain mathematical guarantees that are useful for game design.

2 Limping Polygons

Firstly, we extend the concept of the limping triangle to *limping polygons*, for which a similar n:$n+1$ property holds. Figure 2 (left) shows the 3:4:5 triangle which is both a Pythagorean triangle and a limping triangle.

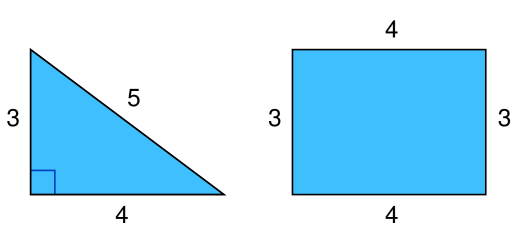

Figure 2. 3:4:5 triangle and 3:4 limping rectangle.

Placing two copies of this triangle together forms the rectangle with sides $\{3, 4, 3, 4\}$ (right). We denote this as a 3:4 limping rectangle, as its sides alternate between 3 and 4 units in length, and shall adopt the notation n:$n+1$ to describe such polygons.

2.1 Concavity and Convexity

The examples shown so far – triangle and rectangle – are *convex* polygons, as each turn is in the same direction. However, it is also possible to have *concave* limping polygons.

Figure 3 (left) shows a concave limping polygon with sides of length $\{1, 2, 1, 2, 1, 2, 1, 2\}$, in which turns alternate left and right with each segment. The polygon shown in Figure 3 (right) with sides of length $\{1, 2, 3, 2, 1, 2, 3, 2\}$ is *not* a limping polygon in the true sense, as each pair of adjacent sides satisfies the n:$n+1$ property but the overall figure includes sides of length $n+2$. The segment lengths of true limping polygons must alternate only between n and $n+1$.

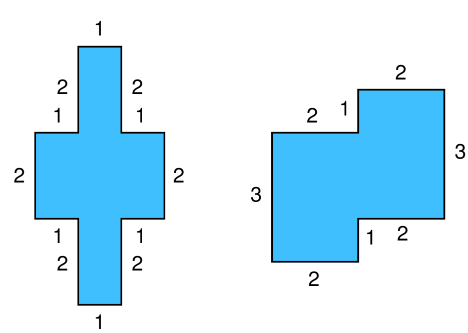

Figure 3. A concave limping polygon (left) and a locally but not globally limping polygon (right).

2.2 Right and Non-Right Polygons

If every turn in a limping polygon is a right angle, then the figure will cover an area that is an even number of square units. This can be seen by observing that side lengths along one axis must all be even while side lengths along the other axis

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must all be odd. Any cross-section along the even axis must pass through an even number of units, and any sum of even numbers will be even, hence the total area of any right-angled limping polygon must be an even number of square units. This can be seen in Figure 4.

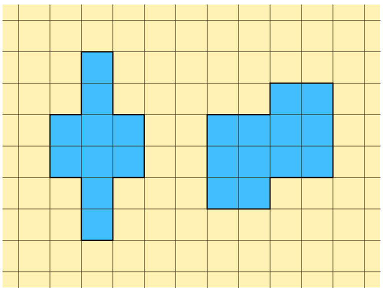

Figure 4. Right limping figures have even area.

Figure 5 shows the first four shapes in the series of minimal (1:2) convex limping polygons, which all have an even number of sides, but in which only the first (rectangular) case has a right-angled basis. Limping polygons must have an even number of sides in order to satisfy the n:n+1 constraint. Hence limping triangles – which inspired the idea – are ironically excluded from the broader umbrella of limping polygons.

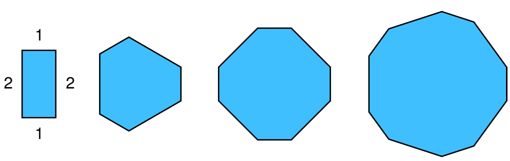

Figure 5. Minimal (1:2) convex limping polygons.

3 Limping Boards

In terms of board design, we are most interested in convex polygons, as they are the most efficient and widely used board shape. More specifically, we are most interested in the first two convex cases shown in Figure 5 – rectangular and irregular hexagonal – as these are the two fundamental shapes that align naturally with the cells of regular square and hexagonal tilings (Figure 6).

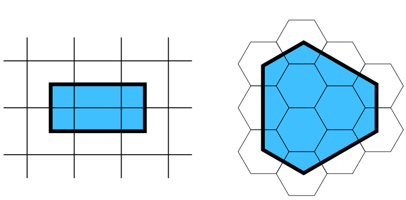

Figure 6. Square and hexagonal bases.

3.1 Square Basis

Limping boards with a square basis are simply $n \times (n+1)$ rectangles. For example, Figure 7 shows a 7:8 limping rectangular board.

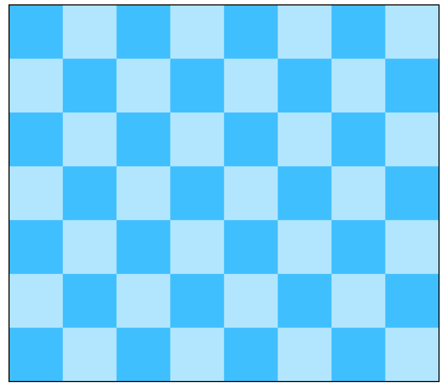

Figure 7. A 7:8 limping rectangular board.

For limping rectangular boards, the number of edge cells C_{e_r} will always be even and is given by:

$$C_{e_r} = 2(2n-1) \tag{1}$$

The total number of cells C_{t_r} will also always be even and is given by:

$$C_{t_r} = n(n+1) \tag{2}$$

Limping rectangular boards are reasonably common, and the BoardGameGeek forum 'Games played on an N x N+1 grid' lists dozens of examples.[1] Note that the terminology $n \times (n+1)$ implies a rectangular shape, whereas the new n:n+1 terminology suggested for limping polygons applies to a wider range of shapes.

3.2 Hexagonal Basis

Limping hexagonal boards, with alternating sides of n and n+1 hexagonal cells, are more interesting for a number of reasons. Figure 8 shows the three smallest limping hexagonal boards, of size 1:2, 2:3 and 3:4. The figures are coloured to show how each can be decomposed into three rhombi of size n^2.

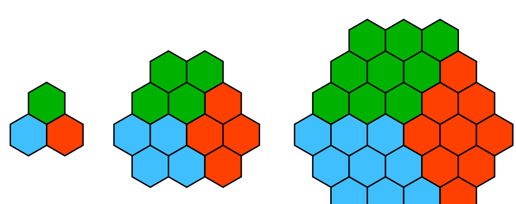

Figure 8. Limping hexagonal boards: 1:2, 2:3, 3:4.

[1] https://boardgamegeek.com/geeklist/23237/games-played-n-n1-grid/

For limping hexagonal boards, the number of edge cells C_{e_h} will always be odd and is given by:

$$C_{e_h} = 3(2n-1) \quad (3)$$

The total number of cells C_{t_h} will have the same parity (i.e. even or odd) as its base size n, and is given by:

$$C_{t_h} = 3n^2 \quad (4)$$

This guaranteed divisibility by three, in both the number of edge cells and total number of cells, makes limping hexagonal boards a promising option for three-player games. It may also have benefits for publishers, if boards can be disassembled into three equal pieces for storage in a game box.

Further, limping hexagonal boards do not have a single central cell, as is the case with the more standard *hexhex board* (i.e. a regular hexagon tessellated by hexagons), but are centred on the intersection of three equally central cells. This is already an interesting design feature which can reduce any inherent first move advantage.[2]

Figure 9 shows the limping rhombus board, which is an unusual hybrid. This board has a hexagonal basis but its shape has more in common with an $n\times(n+1)$ rectangle, giving an even number of edge cells and an even number of total cells. I know of one only game – $n\times(n+1)$ Hex, discussed below – which uses this board shape.

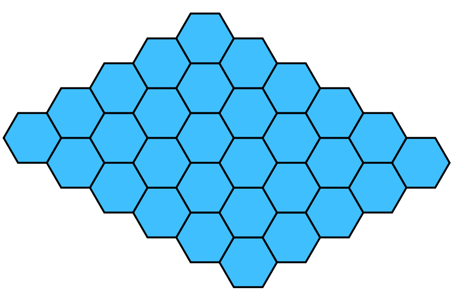

Figure 9. A 5:6 limping rhombus board.

3.3 Cell Parity

The parity of the number of board cells can be an important consideration for some games. For two-player games, it is often desirable to have an even number of cells so that both players have at least the possibility of making an equal number of moves (e.g. Clobber, below). At the same time, an odd number of edge cells can be useful to resolve ties (e.g. Star, below).

Table 1 shows a comparison of edge cell (P_e) and total cell (P_t) parities for the various board shapes and bases discussed above.

	Regular		**Limping**		
	Square	**Hex.**	**Rect.**	**Hex.**	**Rhomb.**
P_e	Even	Even	Even	Odd	Even
P_t	As n	Odd	Even	As n	Even

Table 1. Cell parity of various board types ($n > 1$).

It can be seen that the limping hexagonal board is the only combination that guarantees an even number of total cells (for even sizes of n) but an odd number of edge cells. This, combined with the guaranteed divisibility by 3 and lack of a single central cell, make the limping hexagonal board a perhaps underused resource.

4 Examples

The following examples show instances of games that use limping boards as a basic design feature.

4.1 Clobber

Clobber[3] is a two-player combinatorial game, in which players take turns moving one of their pieces to capture (i.e. clobber) an adjacent enemy piece. The last player to move wins. Figure 10 shows the starting position for a 5:6 game.

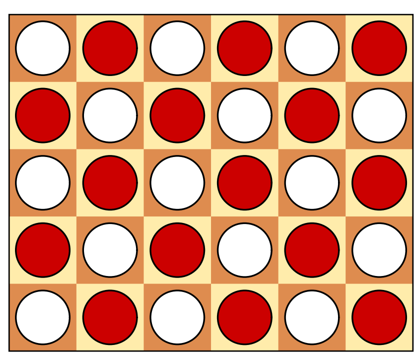

Figure 10. A 5:6 game of Clobber about to start.

Clobber is often played on a 5:6 or other n:$n+1$ limping rectangular board. This guarantees an even number of board cells regardless of n, hence theoretically offers each player an equal number of moves; an important consideration in combinatorial games.

[2]This is one reason why the 14×14 Hex board is often preferred over Piet Hein's original 11×11 board [2].
[3]https://boardgamegeek.com/boardgame/23864/clobber

4.2 Silverman's Minichess

Figure 11 (left) shows the 4×4 game of Minichess invented by mathematician David Silverman.[4]

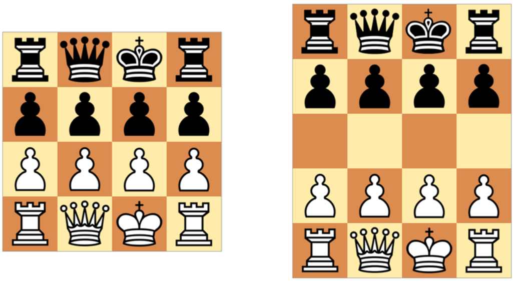

Figure 11. Silverman's 4×4 and 4×5 Minichess.

When a trivial win for White was pointed out, Silverman added an opening rule – Black nominates which pawn White must play first – but this was found to lead to a trivial win for Black. Finally, Silverman solved the trivial win problem by extending the board to a 4:5 limping rectangle with a row of empty cells separating the players' forces, as shown in Figure 11 (right).

This demonstrates a common application of the limping rectangle, as a way of sizing boards between the more standard square sizes. The following considerations often result in limping rectangular boards, although the fact that they are limping is somewhat coincidental in these cases:

- Does the game need a different number of pieces? Add or remove a row/column of pieces.
- Do the opposing forces need more separation? Add a row/column of empty cells.
- Do the opposing forces need to engage more quickly? Remove a row/column of empty cells.

4.3 Bridg-It

The game of Bridg-It, by mathematician David Gale [3], is shown in Figure 12. Players take turns placing a bridge of their colour between available pegs of their colour, in an effort to connect their sides of the board.

The board consists of two orthogonally overlapping 5:6 limping rectangular grids of pegs. The overall board is ostensibly a sparse 11×11 grid of pegs, but each player's moves are constrained to their own 5:6 limping rectangle. Limping rectangles provide the most square non-square rectangular shape, allowing this neat form of overlap.

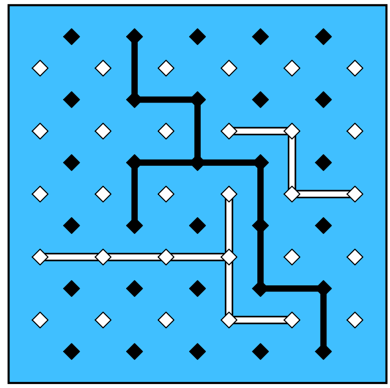

Figure 12. A game of Bridg-It won by Black.

The layout of pegs on the Bridg-It board can also be seen in the starting position of connection game Ayu[5] (Figure 13) as a way to initially separate all pieces, and in the squares of the semiregular truncated square tiling 4.8.8 (Figure 14), although these occurrences are coincidental.

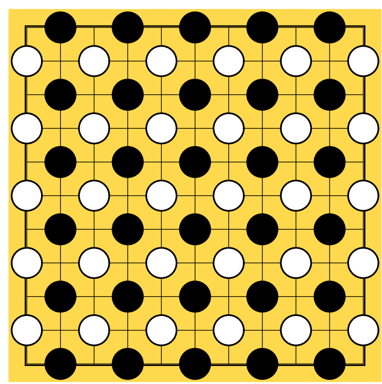

Figure 13. Ayu starting position.

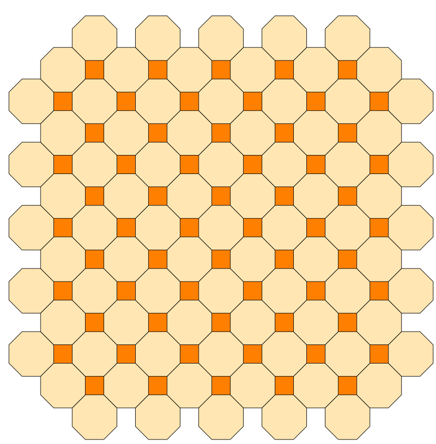

Figure 14. Truncated square tiling 4.8.8.

[4] https://en.wikipedia.org/wiki/Minichess
[5] https://boardgamegeek.com/boardgame/114484/ayu

4.4 Star

Turning now to games on limping hexagonal boards, the game of Star demonstrates their design benefits clearly. Star was designed in 1983 by mathematician Craige Schensted (now known as Ea Ea) [4]. Figure 15 shows a completed 4:5 game won by Black.

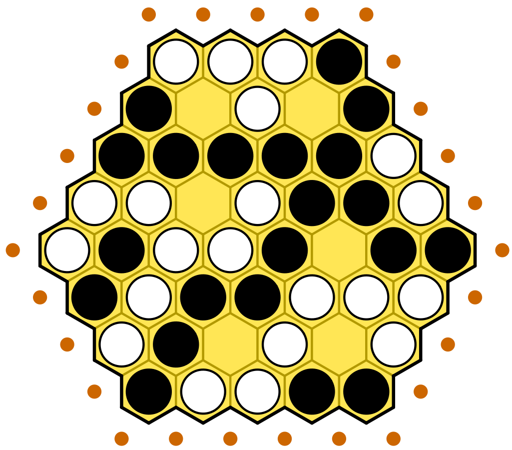

Figure 15. A 4:5 game of Star won by Black.

In Star, two players add a piece of their colour each turn, and win by achieving the highest score at the end of the game. A *star* is a connected chain of same-coloured pieces that contains edge pieces adjacent to at least three external dots (see Figure 15). Each star's score is the number of external dots that its edge pieces are adjacent to minus 2. Each player's score is the total score of their stars.

While the scoring system may initially sound confusing, it is extremely intuitive in play, and the game basically boils down to connecting as many edge cells of your colour into as few groups as possible, to maximise the external neighbour count while minimising the -2 group penalty.

Black has won the game shown with star scores of 11 + 2 = 13 points, compared to White's 3 + 1 + 4 + 4 = 12 points. Black wins despite only having nine pieces on edge (scoring) cells while White has twelve pieces on edge cells; Black's stronger connection reduces the number of black groups and makes the difference.

Star boards with even n (e.g. size 4:5) will have an even number of cells, giving both players the potential for the same number of moves as there is no capture. While games tend to end as soon as it is no longer possible for either player to score any more points, as is the case with the game shown in Figure 15, parity can become critical if the board fills up and the last move proves decisive. Giving the opening player both the first move *and* the last move would be too strong an advantage. Further, the odd number of edge cells means an odd number of total points on offer, hence no game of Star can ever end in a tie once all edge cells are filled.

4.5 Projex

Figure 16 shows the game of Projex, invented by mathematicians Lloyd Shapley and Bill Taylor in 1994.[6] The board 'wraps around' at the edges as shown to form a projective plane, e.g. edge cell *a* is considered adjacent to edge cells *l* and *k*, etc.

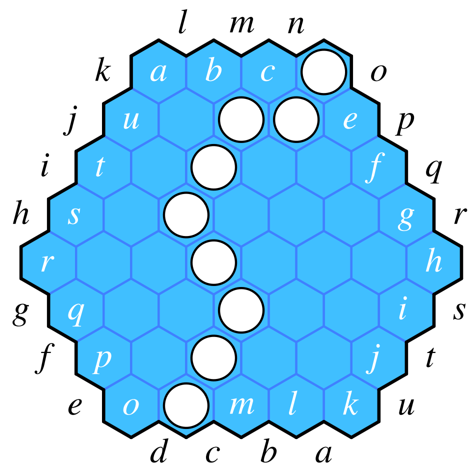

Figure 16. A 4:5 Projex board and a win for White.

Players win at Projex by making a chain of their pieces that forms a non-trivial loop encircling the projective plane. Explaining exactly what this means is beyond the scope of this paper, suffice it to say that players win by forming a chain of their pieces with an odd number of edge crossings, such as the white chain shown in Figure 16 which has a single edge crossing.

Again, even board sizes n give an even number of board cells, guaranteeing an equal number of moves for both players if the board fills up. However, the more fundamental reason that Projex is played on a limping hexagonal board is due to the geometry of projective edge connections, as edges of length n nestle into neighbouring edges of length $n+1$ on the hexagonal grid, and vice versa. To illustrate this point, Figure 17 shows three adjacent limping hexagonal boards packed around a common intersection. Note that the boards are centred relative to each other.

Figure 18 shows three hexhex boards packed together in a similar way. Notice in this case, however, that this packing is *oriented* and can progress either clockwise (left) or anticlockwise (right) around the intersection, which would

[6] https://boardgamegeek.com/boardgame/31805/projex

make wraparound projective adjacency ambiguous. The centred packing allowed by limping hexagonal boards avoids this ambiguity.

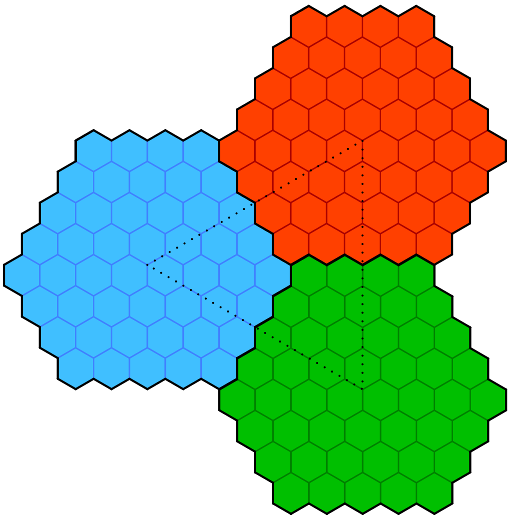

Figure 17. Limping hexagons nestle nicely.

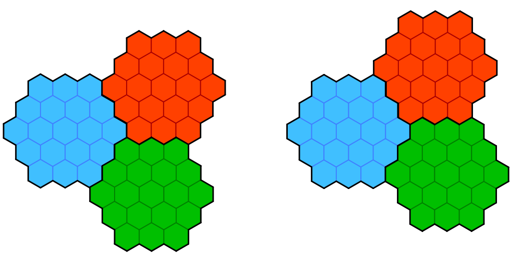

Figure 18. Hexhex board packings are oriented.

4.6 Shoulder to Shoulder

Figure 19 shows the starting position for Shoulder to Shoulder, a game for three players designed by David Parlett in 1975.[7] The game was originally designed on a size 7 hexhex board with 127 cells, but it was found that the central cell gave the player who occupied it a huge advantage. A playtester then suggested reducing the board to that shown – a 6:7 limping hexagonal board – which provided the following benefits:

- No more single central cell.
- Number of edge cells C_{e_h} divisible by 3.
- Total number of cells C_{t_h} divisible by 3.
- Number of cells reduced from 127 to 108.

These properties combined harmoniously to benefit the game. The limping board design allowed an intuitive starting arrangement for each player's twelve pieces as shown, and solved other problems that had been plaguing the game.

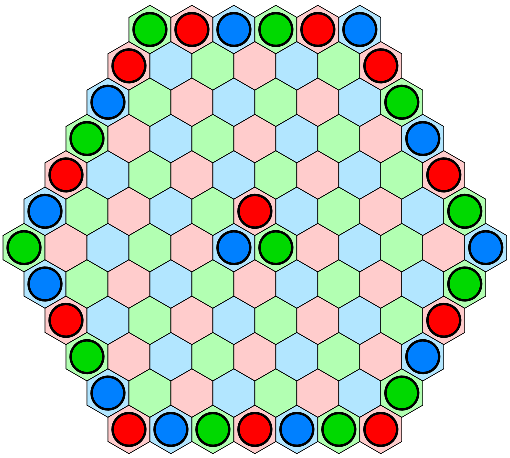

Figure 19. Shoulder to Shoulder starting position.

4.7 Tres

The game of Tres, designed by Fred Horn in 2011 and shown in Figure 20, involves a grid of pegs in a 4:5 limping hexagon.[8] Players slot a three-holed triangle of their colour over three pegs each turn, and aim to surround the greatest number of pins with pieces of their colour. Pieces may stack up to three triangles high.

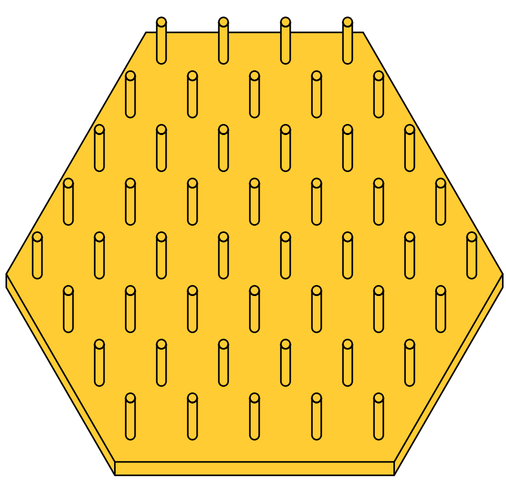

Figure 20. The Tres board.

Horn explains that the Tres design was originally a size-5 hexhex layout, but was reduced to a 4:5 limping hexagon in order to make games

[7] http://www.parlettgames.uk/gamepie/shoulder.html
[8] Personal correspondence from Horn including unpublished Tres specification and rule sheet.

a bit shorter and reduce the amount of material required. An added benefit of the limping shape is that it reduces the board from 6-fold to 3-fold rotational symmetry, which Horn points out provides greater scope for strategy and allows more strategic play earlier in games.

4.8 Zertz

Figure 21 shows the starting layout for Zertz, a GIPF game designed by Kris Burm in 1999 [5]. The 37 black discs show the layout for the standard game, while dotted positions are for additional pieces that can be added for longer games.

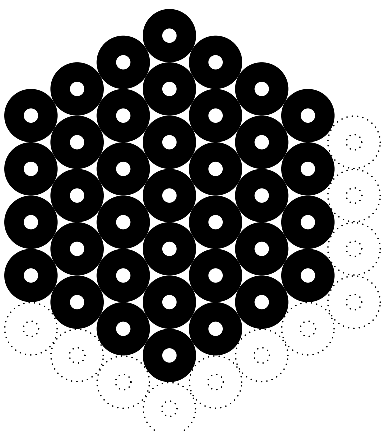

Figure 21. Starting layout for Zertz.

If all 48 discs are used in the extended game, then the starting layout forms a 4:5 limping hexagon. This is another case of extending a standard board in one direction to coincidentally produce a limping board, as per Minichess.

4.9 $n\times(n+1)$ Hex

The game of Hex is typically played on an $n\times n$ rhombus of hexagons [2]. However, Martin Gardner describes an interesting variant played on an $n\times(n+1)$ rhombus of hexagons [6], shown in Figure 22, with the first player aiming to connect the two sides farthest apart and the second player aiming to connect the two sides closest together. This is the only game played on an n:$n+1$ limping rhombus that I am aware of.

The interesting thing about this version of Hex is that it provides a guaranteed win for the second player, due to a point-pairing symmetry strategy. Each time the opponent plays, then playing in the matching cell with the same letter ensures that the closest board sides are connected before the farthest sides.

The partition of the n:$n+1$ limping rhombus into two mirrored triangles, shown in Figure 22, recalls the doubling of limping triangles discussed in Section 2 that started this discussion.

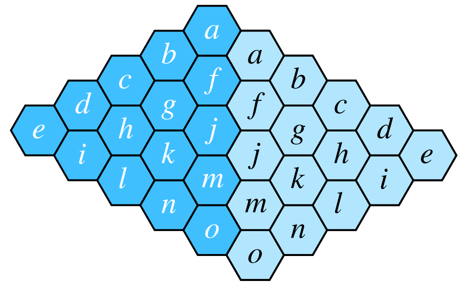

Figure 22. Winning strategy for $n\times(n+1)$ Hex.

This raises the intriguing possibility of similar point-tripling strategies for some three-player games on hexagonal limping boards, e.g. based on the partitions shown in Figure 8. But such strategies would have to take into account intervening moves by *both* opponents so are unlikely.

5 Key Properties for Design

Some key properties in terms of game design, for the three types of limping board discussed above, are as follows:

1. *Limping Rectangular Boards*:
   - Even number of edge cells.
   - Even number of total cells.
   - 2-fold symmetry (rather than 4-fold).
   - Extend a row to separate forces.
   - Remove a row to encourage conflict.

2. *Limping Hexagonal Boards*:
   - Odd number of edge cells.
   - Odd or even number of total cells (as per n).
   - Number of edge cells divisible by 3.
   - Number of total cells divisible by 3.
   - No single central cell.
   - Projective edge crossings are centred.
   - 3-fold symmetry (rather than 6-fold).

3. *Limping Rhombus Boards*:
   - Even number of edge cells.
   - Even number of total cells.
   - Subject to symmetry strategies.

In all cases, the limping property guarantees that certain board sides will be longer or shorter than others, and closer or further from the opposite board sides than others. This asymmetry could be exploited to handicap games as needed. For example, in a connection game, the weaker player – or second player if there is a first move advantage – could aim to connect the longer board edges which are closer together, while their opponent could aim to connect the shorter board edges which are further apart. The combination of more cells closer together gives the weaker player a double advantage to balance things.

6 Conclusion

The extension of the mathematical idea behind limping triangles to limping polygons defines a class of shapes with some useful properties relevant to the design of game boards. Limping rectangular boards (with a square basis) provide useful board sizes in between standard square sizes, and guarantee an even number of board cells regardless of size. Limping hexagonal boards (with a regular hexagonal basis) guarantee an even number of cells when n is even, and an odd number of edge cells regardless of size. They also pack with projected neighbours to provide unambiguous wraparound adjacencies on the hexagonal grid. Other useful properties are listed above.

When developing games, it is worth keeping in mind the properties of the various board types summarised in Table 1. These constraints provide mathematical guarantees that can be exploited to elegantly solve certain game design problems. It is no coincidence that most of the designers mentioned above, who have used limping boards in their games, are mathematicians.

Acknowledgements

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[2] Browne, C., *Hex Strategy: Making the Right Connections*, Wellesley, AK Peters, 2000.

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[5] Thomashow, M., 'Project GIPF: Design and Aesthetics', *Game & Puzzle Design*, vol. 1, no. 1, 2015, pp. 75–79.

[6] Gardner, M., 'The Game of Hex', in *Hexaflexagons and Other Mathematical Diversions: The First Scientific American Book of Puzzles & Games*, Chicago, Chicago University Press, 1959, pp. 73–83.

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Gadeiro Challenge #4

Pack the pieces on the right to fill the shape on the left. Gadeiro is described on pages 39–41.